Exercise 1: Diffusive noise (stochastic spike arrival)

Consider a passive membrane receiving stochastic synaptic input \( S(t) = \sum_f \delta(t - t^f_k) \), where the index \( f \) runs over the firing times of a presynaptic neuron. The spike train starts only at \( t = 0 \), so that \( t^f_k > 0 \) for all firing times. The membrane potential obeys the equation:

\[
\frac{du}{dt} = -\frac{u - u_{\text{rest}}}{\tau} + \frac{q}{C} S(t),
\]

where \( q \) is the charge brought by each spike and \( C \) is the capacitance of the membrane. The solution to this equation writes:

\[
u(t) = u_{\text{rest}} + q R \int_0^t \exp(-s/\tau)S(t-s)ds
\]

1.1 Calculate the expected voltage \( \langle u(t) \rangle \) as a function of \( t \) for a constant presynaptic rate \( \langle S(t) \rangle = \nu \) for \( t \geq 0 \) (\( \nu = 0 \) for \( t < 0 \)). Where \( \langle \rangle \) is the average over multiple repetitions or over a population of neurons having the same dynamics and inputs.

1.2 Calculate \( \langle u(t)^2 \rangle \). Assume that the spike times of the presynaptic neuron are uncorrelated, i.e., \( \langle S(t)S(t') \rangle = \nu \delta(t - t') + \nu^2 \), and use Eq. 2.

1.3 Calculate the variance of the potential across multiple repetitions: \( \text{Var}[u(t)] = \langle [u(t) - \langle u(t) \rangle]^2 \rangle \).

Homework:

1.4 Calculate the autocorrelation of the voltage \( \langle u(t)u(t') \rangle \) in the steady state regime (replace the upper bound of the integral by \( \infty \) in equation 2).

1.5 Suppose that there are two presynaptic neurons which fire independently with rates \( \nu_1 = \langle S_1(t) \rangle \) and \( \nu_2 = \langle S_2(t) \rangle \), such that the input to the postsynaptic neuron is given by \( w_1 S_1(t) + w_2 S_2(t) \) where \( w_i \) denote the synaptic weights. Calculate again the mean and autocorrelation of the voltage.

1.6 Redo question 1.5 with correlated spike trains \( S_1(t) = S_2(t) \).
Exercise 2: Firing statistics

Consider a stochastic spike generation process in discrete time. The probability of generating a spike in a time $\Delta t$ is $P_{\Delta t} = \nu \Delta t$. Hence when we take the limit of $\Delta t$ to 0 the expected value of the quantity $S(t) = \sum_f \delta(t - t^{(f)}_k)$ is:

$$\langle S(t) \rangle = \lim_{\Delta t \to 0} \frac{P_{\Delta t}(t)}{\Delta t} = \nu ; \text{ for } t > 0.$$

Consider the probability of having two spikes in different time bins around $t$ and $t'$. Define $\langle S(t)S(t') \rangle$ in a similar fashion, and show that it is equal to $\nu \delta(t - t') + \nu^2$.

Exercise 3: Coding by spikes

Imagine we are injecting a step current in a neuron that receives no other input. In this exercise we investigate how the time of the first spike $T$ codes for the amplitude of the step current $I_0$. You should interpret your results in terms of coding efficiency: How is the time of the first spike coding for the current amplitude in each case? How could you make a precise measurement of the current amplitude with each model?

3.1 Determine the timing of the first spike as a function of $I_0$ for a leaky integrate-and-fire model.

3.2 Consider a poisson neuron with a firing rate proportional to current: $\rho(t) = kI(t)$. Can you determine the time of the first spike? Make a qualitative sketch of the probability distribution of the time of the first spike. Calculate the expected $T(I_0)$.

3.3 Consider leaky integrator neuron model:

$$\frac{du}{dt} = \frac{-u}{\tau} + \frac{I(t)}{C}$$

where $u$ is the membrane potential above rest. Consider further that firing is stochastic and occurs via escape noise proportional to the potential. Precisely, the instantaneous probability to fire is given by:

$$\rho(t) = \begin{cases} u(t) - a & \text{for } u(t) > a \\ 0 & \text{for } u(t) < a. \end{cases}$$

where $a$ is some threshold potential above which the neuron has a non-zero probability to spike. Can you determine the time of the first spike? Make a qualitative sketch of the probability distribution of the time of the first spike. Calculate the expected $T(I_0)$ in the limit of very large $\tau$ using the concepts from renewal theory (lecture 8).