Biological Modeling of Neural Networks

Week 5

NETWORKS of NEURONS and ASSOCIATIVE MEMORY

Wulfram Gerstner
EPFL, Lausanne, Switzerland

5.1 Introduction
- networks of neuron
- systems for computing
- associative memory

5.2 Classification by similarity

5.3 Detour: Magnetic Materials

5.4 Hopfield Model

5.5 Learning of Associations

5.6 Storage Capacity
Systems for computing and information processing

**Brain**

- Distributed architecture
- \(10^{10}\) proc. Elements/neurons
- No separation of processing and memory

**Computer**

- Von Neumann architecture
- 1 CPU
- \(10^{10}\) transistors

- CPU
- memory
- input
Systems for computing and information processing

- 10,000 neurons
- 3 km wire

1 mm
Systems for computing and information processing

Brain

10,000 neurons
3 km wire

Distributed architecture
$10^{10}$ neurons
$10^{4}$ connections/neurons

No separation of processing and memory
Associations, Associative Memory

Read this text NOW!

This means that you are able to fill in missing info.
pattern completion/word recognition

Your brain fills in missing information: ‘associative memory’
Week 5: Networks of Neurons - Introduction

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5.2 Classification by similarity: **pattern recognition**

Classification: comparison with prototypes

**Noisy image**

**Prototypes**
5.2 Classification by similarity: pattern recognition

Classification by closest prototype

\[ |x - p^T| \leq |x - p^A| \]

Noisy image

Prototypes

Blackboard:
5.2 pattern recognition and Pattern completion

Aim: Understand Associative Memory

Noisy image → Associative memory/collective computation → Full image

Partial word → Brain-style computation → Full word
A typical neuron in the brain makes connections
- To 6-20 neighbors
- To 100-200 neurons nearby
- To more than 1000 neurons nearby
- To more than 1000 neurons nearby or far away.

In a typical crystal in nature, each atom interacts
- with 6-20 neighbors
- with 100-200 neurons nearby
- with more than 1000 neurons nearby
- with more than 1000 neurons nearby or far away.
Biological Modeling of Neural Networks

Week 5: Networks of Neurons - Introduction

√ 5.1 Introduction
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5.3 Detour: magnetism
5.3 Detour: magnetism

Noisy magnet  →  pure magnet
5.3 Detour: magnetism

Elementary magnet

\[ S_i = +1 \]
\[ S_i = -1 \]

**Blackboard: example**

dynamics

\[ S_i(t + 1) = \text{sgn}\left[ \sum_j S_j(t) \right] \]

Sum over all interactions with \( i \)
5.3 Detour: magnetism

Anti-ferromagnet

Elementary magnet

\[ S_i = +1 \]
\[ S_i = -1 \]
\[ w_{ij} = +1 \]
\[ w_{ij} = -1 \]

Dynamics

\[ S_i(t + 1) = \text{sgn}\left[ \sum_j w_{ij} S_j(t) \right] \]

Sum over all interactions with i
5.3 Magnetism and memory patterns

Elementary pixel

- $S_i = +1$
- $S_i = -1$

$w_{ij} = +1$
- $w_{ij} = -1$

Dynamics

$$S_i(t+1) = \text{sgn}[\sum_j w_{ij}S_j(t)]$$

Hopfield model: Several patterns $\rightarrow$ next section

Sum over all interactions with $i$
Exercise 1: Associative memory (1 pattern)

Elementary pixel

- $S_i = +1$
- $S_i = -1$
- $w_{ij} = +1$

Dynamics

$$S_i(t+1) = \text{sgn} \left[ \sum_j w_{ij} S_j(t) \right]$$

9 neurons
- define appropriate weights
- what happens if one neuron wrong?
- what happens if $n$ neurons wrong?

Next lecture at 10h15
Week 5: Networks of Neurons-Introduction

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5.4 Hopfield Model of Associative Memory

Prototype $\overrightarrow{p}^1$

Prototype $\overrightarrow{p}^2$

Hopfield model

Interactions

$$w_{ij} = \sum_{\mu} p_i^\mu p_j^\mu$$

Sum over all prototypes

Dynamics

$$S_i(t+1) = \text{sgn}\left[ \sum_{j} w_{ij} S_j(t) \right]$$

Sum over all interactions with $i$
5.4 Hopfield Model of Associative Memory

**Prototype**

$\bar{p}^1$

**Random patterns, fully connected: Hopfield model**

**Dynamics**

$S_i(t + 1) = \text{sgn} \left[ \sum_j w_{ij} S_j(t) \right]$

This rule is very good for **random** patterns. It does not work well for correlated patterns.

**Prototype**

$\bar{p}^1$

**Dynamics**

$w_{ij} = \sum_{\mu} p_{i}^{\mu} p_{j}^{\mu}$  

Sum over all prototypes.
5.4 Hopfield Model of Associative Memory

\[ S_i(t + 1) = \text{sgn}\left[ \sum_j w_{ij} S_j(t) \right] \]

\[ w_{ij} = \sum_\mu p^\mu_i p^\mu_j \]

\[ m^\mu(t) = \frac{1}{N} \sum_j \xi^\mu S_j(t) \]

\[ m^\mu(t + 1) = \frac{1}{N} \sum_j \xi^\mu S_j(t + 1) \]
5.4 Hopfield Model of Associative Memory

Prototype

\( \overline{p^1} \)

*Finds the closest prototype*

\( \text{i.e. maximal overlap} \)

*\( m_{\mu} \) (similarity)*

Hopfield model

Interacting neurons

**Computation**

- without CPU,
- without explicit memory unit
Assume 4 patterns. At time $t=0$, overlap with Pattern 3, no overlap with other patterns. Discuss temporal evolution (of overlaps) (assume that patterns are orthogonal).

Prototype $\vec{p}^1$

$$w_{ij} = \frac{1}{N} \sum_{\mu} p_{i}^{\mu} p_{j}^{\mu}$$

$$S_i(t+1) = \text{sgn}[\sum_{j} w_{ij} S_j(t)]$$

Sum over all interactions with i

Next lecture at 11h15
5.1 Introduction
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5.5 Learning of Associations

5.6 Storage Capacity
5.5 Learning of Associations

Where do the connections come from?

Hebbian Learning

When an axon of cell $j$ repeatedly or persistently takes part in firing cell $i$, then $j$’s efficiency as one of the cells firing $i$ is increased

- local rule
- simultaneously active (correlations)

Hebb, 1949
5.5 Hebbian Learning of Associations
5.5 Hebbian Learning of Associations

item memorized
5.5 Hebbian Learning: Associative Recall

Recall:
Partial info

item recalled
Tell me the **shape** for the following list of 5 items:

![Object shapes](image)

Tell me the **color** for the following list of 5 items:

![Object colors](image)

be as fast as possible:

![Time](image)
Tell me the **color** for the following list of 5 items:

Red
Blue
Yellow
Green
Red

**Stroop effect:**
Slow response: hard to work
Against natural associations
Hierarchical organization of
Associative memory

animals

birds

fish

Name as fast as possible

an example of a bird

swan (or goose or raven or ...)

Write down first letter: s for swan or r for raven ...
5.5 Associative Recall

Nommez au plus vite possible un exemple d’un / d’une name as fast as possible an example of a
outil tool
couleur color
fruit fruit
instrument music
de musique instrument

Associative Recall
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learning of prototypes

Prototype $\vec{p}^1$

Prototype $\vec{p}^2$

Q; How many prototypes can be stored?

interactions

$$w_{ij} = \frac{1}{N} \sum_{\mu} p_i^\mu p_j^\mu$$

Sum over all prototypes

$$(1)$$

$$S_i(t + 1) = \text{sgn}[\sum_j w_{ij} S_j(t)]$$

dynamics

all interactions with $i$
Q; How many prototypes can be stored?

**Prototype**

\[ \mathbf{p}^1 \]

\[ \mathbf{p}^2 \]

**Dynamics (2)**

**Minimal** condition: pattern is fixed point of dynamics
- Assume we start directly in one pattern
- Pattern stays

Attention: Retrieval requires more (pattern completion)

Random patterns

**Interactions (1)**

\[ w_{ij} = \sum_{\mu} p_{i\mu}^\mu p_{j\mu}^\mu \]

\[ S_i(t+1) = \text{sgn}[\sum_j w_{ij} S_j(t)] \]
Exercise 4 now: Associative memory

Q: How many prototypes can be stored?

Random patterns \(\rightarrow\) random walk

a) show relation to erf function: importance of \(p/N\)

b) network of 1000 neurons – allow at most 1 wrong pixel?

c) network of \(N\) neurons – at most 1 promille wrong pixels?

Interactions (1)

\[ w_{ij} = \sum_{\mu} p_{i}^{\mu} p_{j}^{\mu} \]

Dynamics (2)

\[ S_i(t + 1) = \text{sgn} \left[ \sum_{j} w_{ij} S_j(t) \right] \]
Prototype $p^1$

Prototype $p^2$

Random patterns

Interactions (1) $w_{ij} = \sum_{\mu} p_i^\mu p_j^\mu$

Dynamics (2)

$S_i(t+1) = \text{sgn} \left[ \sum_j w_{ij} S_j(t) \right]$  

**Minimal** condition: pattern is fixed point of dynamics
- Assume we start directly in one pattern
- Pattern stays

Attention: Retrieval requires more (pattern completion)
Q; How many prototypes can be stored?
Biological Modeling of Neural Networks

Week 6

Hebbian LEARNING and ASSOCIATIVE MEMORY

Wulfram Gerstner
EPFL, Lausanne, Switzerland

6.1 Stochastic Hopfield Model
6.2. Energy landscape
6.3. Low-activity patterns
6.4. Attractor memory
- spiking neurons
- experimental data
6.1 Review: Hopfield model

\[ w_{ij} = \frac{1}{N} \sum_{\mu} p^\mu_i p^\mu_j \]

Sum over all prototypes

\[ S_i(t + 1) = \text{sgn} \left[ \sum_j w_{ij} S_j(t) \right] \]
## 6.1 Stochastic Hopfield model

Prototype $\vec{p}^1$

Prototype $\vec{p}^2$

Random patterns

Interactions (1) $w_{ij} = \sum_\mu p_i^\mu p_j^\mu$

Dynamics (2)

\[
\Pr\{S_i(t+1) = +1 | h_i\} = g[h_i] = g\left[\sum_j w_{ij} S_j \cdot t\right]
\]

\[
\Pr\{S_i(t+1) = +1 | h_i\} = g\left[\sum_\mu p_i^\mu m^\mu \cdot t\right]
\]
6.1 Stochastic Hopfield model: memory retrieval

\[ m^\nu(t + \Delta t) \]

\[ m^\nu(t_0) \]

\[ m^\nu(t) \]
6.1 Stochastic Hopfield model: memory retrieval

\[ m^3 = 1 \]

\[ m^{17} = 1 \]
6.1 Stochastic Hopfield model

Dynamics (2)

\[
\Pr\{S_i (t+1) = +1 | h_i \} = g[h_i] = g \left[ \sum_j w_{ij} S_j \right]
\]

\[
\Pr\{S_i (t+1) = +1 | h_i \} = g \left[ \sum_\mu p_\mu^i m^\mu \right]
\]

Assume that there is only overlap with pattern 17:

two groups of neurons: those that should be ‘on’ and ‘off’

\[
\Pr\{S_i (t+1) = +1 | h_i = h^+ \} = g \left[ m^{17} \right]
\]

\[
\Pr\{S_i (t+1) = +1 | h_i = h^- \} = g \left[ -m^{17} \right]
\]

\[
2m^{17}(t+1) = g \left[ m^{17} \right] + \{1 - g \left[ -m^{17} \right]\} - g \left[ m^{17} \right] - \{1 - g \left[ -m^{17} \right]\}
\]
6.1 Stochastic Hopfield model: memory retrieval

\[ 2m^{17}(t+1) = g\left[ m^{17} \ t \right] + \{1 - g\left[ -m^{17} \ t \right]\} - g\left[ m^{17} \ t \right] - \{1 - g\left[ -m^{17} \ t \right]\} \]

\[ m^{17}(t+1) = F\left[ m^{17} \ t \right] \]
Week 6: Hopfield model continued

6.1 Stochastic Hopfield Model
6.2. Energy landscape
6.3. Low-activity patterns
6.4. Attractor memorie
   - spiking neurons
   - experimental data
6.2 memory retrieval

$m^3 = 1$  
$m^{17} = 1$
6.2 Symmetric interactions: Energy picture

\[ m^3 = 1 \]

\[ m^{17} = 1 \]
Exercise 2 now: Energy picture

\[ S_i(t + 1) = \text{sgn}\left[ \sum_j w_{ij} S_j(t) \right] \]

\[ E = -\sum_i \sum_j w_{ij} S_i S_j \]

\[ m^3 = 1 \]

\[ m^{17} = 1 \]
Week 6: Hopfield model continued

Biological Modeling of Neural Networks

Week 6
Hebbian LEARNING and ASSOCIATIVE MEMORY

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EPFL, Lausanne, Switzerland

6.1 Stochastic Hopfield Model
6.2. Energy landscape
6.3. Low-activity patterns
6.4. Attractor memories
- spiking neurons
- experimental data
6.3 Attractor memory

$m^3 = 1$

$m^{17} = 1$
Memory with spiking neurons

- Mean activity of patterns?
- Separation of excitation and inhibition?
- Modeling?
- Neural data?
6.3 attractor memory with low activity patterns

Random patterns +/-1 with zero mean $\rightarrow$
50 percent of neurons should be active in each pattern

$$w_{ij} = \frac{1}{N} \sum_{\mu} p_i^\mu p_j^\mu$$
Random patterns +/-1 with low activity (mean = $a<0$) → 20 percent of neurons should be active in each pattern

$$w_{ij} = \frac{1}{N} \sum_{\mu} (p_i^{\mu} - b)(p_j^{\mu} - a)$$

Some constant activity
Biological Modeling of Neural Networks

Week 6
Hebbian LEARNING and ASSOCIATIVE MEMORY

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6.1 Stochastic Hopfield Model
6.2. Energy landscape
6.3. Low-activity patterns
6.4. Attractor memories
   - spiking neurons
   - experimental data
6.4 attractor memory with spiking neurons

Hebb-rule:
Active together

\[ w_{ij} = \frac{1}{N} \sum_{\mu} (p_i^\mu + 1)(p_j^\mu + 1) \]
Overlap with patterns 1 … 3
Overlap with patterns 1 … 11 (80 patterns stored!)
Memory with spiking neurons

- Low activity of patterns?
- Separation of excitation and inhibition?
- Modeling?

- Neural data?

All possible
6.4 memory data

Human Hippocampus

Delayed Matching to Sample Task

Animal experiments

Sample → 1s → Match

Sample → 1s → Match
In the Hopfield model, neurons are characterized by a binary variable $S_i = +/1$. For an interpretation in terms of spikes it is, however, more appealing to work with a binary variable $x_i$ which is zero or 1.

(i) Write $S_i = 2x_i - 1$ and rewrite the Hopfield model in terms of $x_i$. What are the conditions so that the input potential is

$$h_i = \sum_j w_{ij} x_j$$

(ii) Repeat the same calculation for low-activity patterns and weights

$$w_{ij} = \frac{1}{N} \sum_{\mu} (p_i^\mu - b)(p_j^\mu - a)$$

with some constants $a$ and $b$.
The end